Centres de gravetat

- M amb puntuals (sistema discret)

$$\vec{r}' = \frac{\sum m_i \vec{r}_i}{\sum m_i} \quad \text{on } \vec{r}' = (\bar{x}, \bar{y}, \bar{z})$$

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

- M amb contínua

$$\bar{x} = \frac{\int x \, dm}{\int dm} \longrightarrow \begin{cases} dm = \lambda \, dl \\ dm = \sigma \, dA \\ dm = \rho \, dV \end{cases} \longrightarrow \bar{x} = \frac{\int x \, dl}{\int dl} = \frac{Q_3}{l}$$

$$\bar{x} = \frac{\int x \, dA}{\int dA} = \frac{Q_3}{A}$$

(ρ ctte)

- Cons composta:

$$\bar{x} = \frac{\sum A_i \cdot \bar{x}_i}{\sum A_i}$$

Pappus - Guldinus

$$S_{rev} = 2\pi \bar{x} \, l \quad (\text{si } \gamma \text{ és sin rev})$$

$$V_{rev} = 2\pi \bar{x} A$$

$$V_{rev} = h \cdot S_{rev} \quad \text{(si } \gamma \text{ és rev)}$$

Forces distribuïdes

$$F_i = \int w(x) \, dx = \int dA = A$$

$$x_i = \int x \, w_i(x) \, dx$$

$$M = dP$$

Casos útils: + Eq recta → $y = mx + b = \frac{y_0}{x_0} x + b$

* Eq circum → $R^2 = (x - x_0)^2 + (y - y_0)^2$ (centre)

* Integrals per parts → $\int u \, dv = uv - \int v \, du$

$$\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du$$

* Integrals de línies → $\int dl = \int |\vec{r}\,'(x,y)| \, dx$ (mòdul de la derivada)

$\vec{r}\,'(x,y) = (x, f(x))$ Derivada →

* $\int x \sqrt{a - x^2} = \frac{1}{\frac{1}{2}+1} \frac{1}{(-2)} (a - x^2)^{\frac{1}{2}+1}$

Moments d'inèrcia de seccions

Moments d'inèrcia respecte [L]⁴ (sempre > 0)

$$I_x = \int y^2 \, dA$$

$$I_y = \int x^2 \, dA$$

$$J_O = \int r^2 \, dA = \int (x^2 + y^2) \, dA = I_x + I_y$$

↳ moment polar d'inèrcia (respecte punt O o respecte eix z)

Radi de gir $\quad i_{Gx} = \sqrt{\frac{I_{Gx}}{A}} \quad [L]$

Producte d'inèrcia [L⁴] ($\gtrless 0$)

$$I_{xy} = \int x y \, dA \quad [L^4]$$

−	+
+	−

Simetria respecte $\longleftrightarrow$ $I_{xy} = 0$ ⇓ DPI (direcció principal d'inèrcia)

Simetria implica que és DPI, però totes DPI no implica simetria respecte totes les eixos

On és les inèrcies màximes? → A les eixos centrals

Teorema Steiner

↳ I del centre de masses

$$I_x = \bar{I}^c_{x'} + A \, d^2 = I^c_{x'} + A \, \bar{y}^2$$

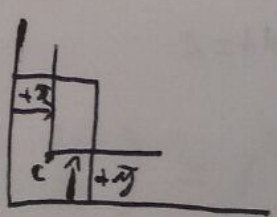
↳ dist entre eixos

$$I_{xy} = \bar{I}^c_{xy} + A \, \bar{x} \, \bar{y}$$

↳ signe !!

$$J_O = \bar{J}_o + A \, (\bar{x}^2 + \bar{y}^2)$$

$$i_y^2 = \bar{i}_o^2 + \bar{y}$$



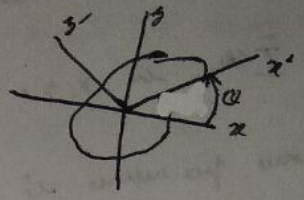
Canvi de referència

$$I'_x = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos(2\alpha) - I_{xy}\sin(2\alpha)$$

$$I'_y = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos(2\alpha) + I_{xy}\sin(2\alpha)$$

$$I'_{xy} = \frac{I_x - I_y}{2}\sin(2\alpha) + I_{xy}\cos(2\alpha)$$

$$I'_x + I'_y = I_x + I_y = J_0$$



Moments d'inèrcia i prod. d'inèrcia d'àrees compostes

$$I_x = \sum I_{x_i}$$

$$J_0 = \sum J_{0_i}$$

$$I_{xy} = \sum I_{xy_i}$$

* Si una àrea té un eix de simetria respecte x o y, el seu producte d'inèrcia és nul

Eixos principals d'inèrcia (EPI)

Eixos respecte als quals els moments d'inèrcia màxim i mínim (moments principals d'inèrcia)

A l'eixos principals $\rightarrow I_{xy} = 0$

Si un eix és de simetria, és un EPI

Un punt principal és aquell que té més d'una parella d'EPI's

Tensor d'inèrcia de secció $\rightarrow$

$$I = \begin{pmatrix} I_x & I_{xy} \\ I_{yx} & I_y \end{pmatrix} = \begin{pmatrix} I_x & I_{xy} \\ I_{xy} & I_y \end{pmatrix}$$

Metodologia per trobar els EPI i els MPI

1. Identificar figures senzilles (□, ○, △ ...)
2. Buscar $\bar{I}^{cg}$ en x, y i z de cadascuna de les figures
3. T. Steiner per repetir el I al punt de càlcul

$$I^p = \bar{I}^{cg} + d^2 S$$

4. Sumar totes els moments d'inercia I_i^p per x, y i z

↳ angle dels EPI

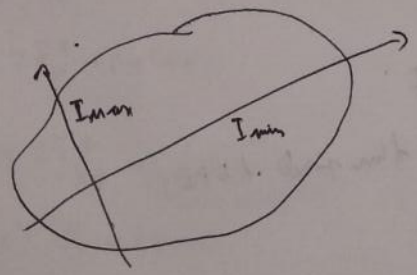
5. $\alpha_p = \frac{1}{2} \operatorname{arctg}\left(-\frac{2I_{xy}}{I_x + I_y}\right)$

$$I_{\substack{min \\ max}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2}$$

a lee: No vemos las fórmulas de cinco pinos, a ciri se quien sabe si el I max o pues el I min

I_{min} → suporta ↓ ↓ deformacions

I_{max} → suporta ↑ ↑ des



Moments d'inèrcia de masses

$F = m \cdot a$ (a: lineal)

$M = \bar{I} \cdot a$ (a: angular; $\bar{I}$: magnitud inercial de massa)

$E_{c_T} = \frac{1}{2} m v^2$

$E_{c_R} = \frac{1}{2} I \omega^2$

$dm = \rho \, dV = \rho \, dx \, dy \, dz$

$m = \rho V$

Moments d'inèrcia de massa $[M][L]^2$ (sempre > 0)

$$I_x = \int (y^2 + z^2) \, dm$$

$$I_y = \int (x^2 + z^2) \, dm$$

$$I_z = \int (x^2 + y^2) \, dm$$

Productes d'inèrcia $[M][L]^2$

$$I_{xy} = \int xy \, dm$$

$$I_{xz} = \int xz \, dm$$

$$I_{yz} = \int yz \, dm$$

Radi de gir

$I_x = k_x^2 m \rightarrow k_x = \sqrt{\frac{I_x}{m}}$ $[L]$

$I_y = \ldots$

$I_z = \ldots$

* El producte d'inèrcia és zero quan almenys un dels dos plans és de simetria.

T. Steiner

$$I_x^P = \bar{I}_x^{CG} + (y^2 + z^2) m$$

$$I_y^P = I_y^{CG} + (x^2 + z^2) m$$

$$I_z^P = I_z^{CG} + (x^2 + y^2) m$$

$$k_x^2 = \bar{k}_{x'}^2 + (y^2 + z^2)$$

$$I_{xy}^P = \bar{I}_{xy}^{CG} + \bar{x}\,\bar{y}\, m$$ (signe!!)

$$I_{xz}^P = \bar{I}_{xz}^{CG} + \bar{x}\,\bar{z}\, m$$

$$I_{yz}^P = \bar{I}_{yz}^{CG} + y z m$$

Plans on, si les línies és simètrica, el producte d'inèrcia és zero:

$I_{xy} \rightarrow xz / yz$

$I_{xz} \rightarrow xy / zy$

$I_{yz} \rightarrow yx / zx$

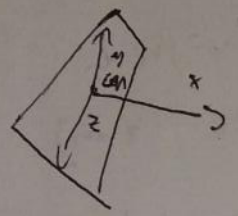
Moments d'inèrcia en plaques fines:

$I_{mass} = \rho \cdot \Delta h \cdot I_{superf} = \rho \frac{m}{A} I_{superf}$ → $I_{x_m} = \rho \cdot h \cdot I_{x_{sup}}$ (con $\frac{m}{A} = \sigma$)

↳ amplada

$I_{x_m} + I_{y_m} = I_{z_m}$

Productes d'inèrcia plaques planes



z i y paral·lels entre si
x perpendicular
(z està al CM)

→ $I_{yz_m} = \rho h I_{yz\,sup}$

$I_{xy_m} = I_{xz_m} = 0$

Tensor d'inèrcia

$$I = \begin{pmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{pmatrix}$$

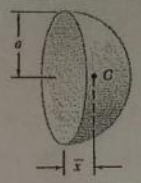
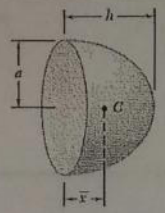
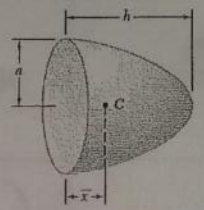
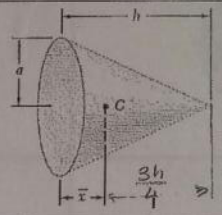
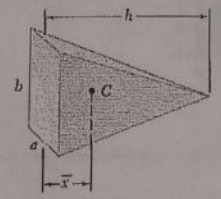
Forma		$\bar{x}$	Volumen
Semiesfera		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semielipsoide de revolución		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloide de revolución		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cono		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pirámide		$\frac{h}{4}$	$\frac{1}{3}abh$

Figura 5.21 Centroides y volúmenes comunes.

medio cono

$\bar{x}$ es la misma

$V = \frac{1}{6}\pi a^2 h$

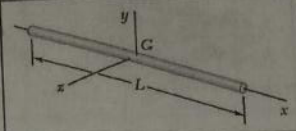
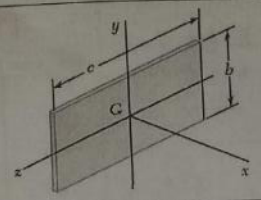
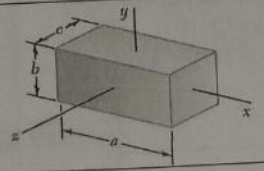
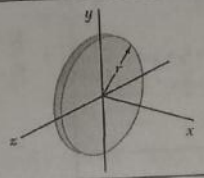
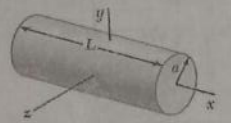
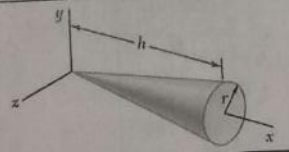
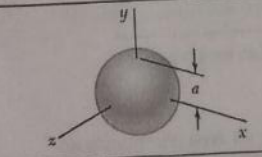
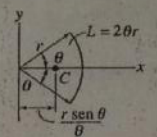
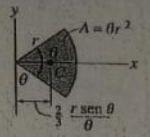
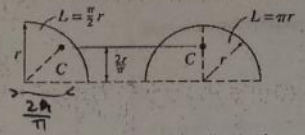
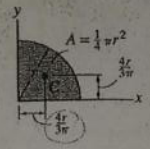
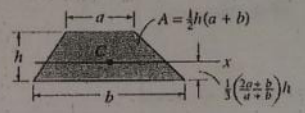
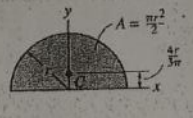
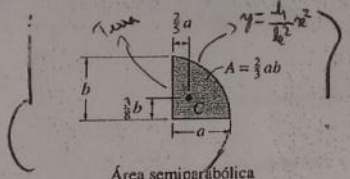
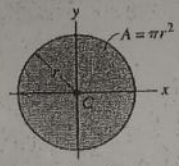
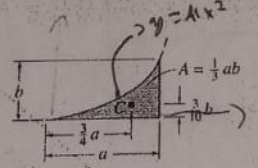
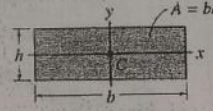
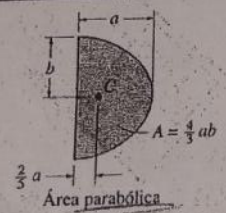
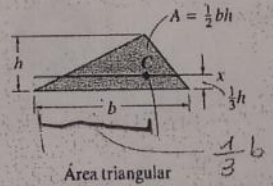
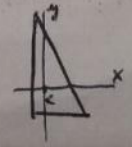
Barra delgada		$I_y = I_z = \frac{1}{12}mL^2$
Placa rectangular delgada		$I_x = \frac{1}{12}m(b^2 + c^2)$ $I_y = \frac{1}{12}mc^2$ $I_z = \frac{1}{12}mb^2$
Prisma rectangular		$I_x = \frac{1}{12}m(b^2 + c^2)$ $I_y = \frac{1}{12}m(c^2 + a^2)$ $I_z = \frac{1}{12}m(a^2 + b^2)$
Disco delgado		$I_z = \frac{1}{2}mr^2$ $I_y = I_x = \frac{1}{4}mr^2$
Cilindro circular		$I_z = \frac{1}{2}ma^2$ $I_y = I_x = \frac{1}{12}m(3a^2 + L^2)$
Cono circular		$I_z = \frac{3}{10}ma^2$ $I_y = I_x = \frac{3}{5}m(\frac{1}{4}a^2 + h^2)$
Esfera		$I_x = I_y = I_z = \frac{2}{5}ma^2$

Figura 9.28 Momentos de inercia de masa de formas geométricas comunes.

www.FreeLibros.me

Propiedades geométricas de líneas y elementos de área

Ubicación del centroide	Ubicación del centroide	Momento de inercia del área
 Segmento de arco circular	 Área de sector circular	$I_x = \frac{1}{4}r^4(\theta - \frac{1}{2}\text{sen } 2\theta)$ $I_x = \frac{1}{4}r^4(\theta + \frac{1}{2}\text{sen } 2\theta)$
 Arcos de un cuarto y de medio círculo	 Área de un cuarto de círculo	$I_x = \frac{1}{16}\pi r^4$ $I_y = \frac{1}{16}\pi r^4$
 Área de un trapecio	 Área de un semicírculo	$I_x = \frac{1}{8}\pi r^4$ $I_y = \frac{1}{8}\pi r^4$
 Área semiparabólica	 Área circular	$I_x = \frac{1}{4}\pi r^4$ $I_y = \frac{1}{4}\pi r^4$
 Área exparabólica	 Área rectangular	$I_x = \frac{1}{12}bh^3$ $I_y = \frac{1}{12}hb^3$
 Área parabólica	 Área triangular	$I_x = \frac{1}{36}bh^3$



$I_{xy} = -\frac{1}{72}k^2a^2$

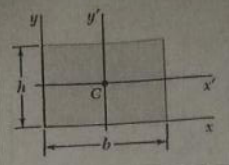
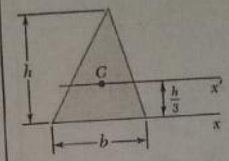
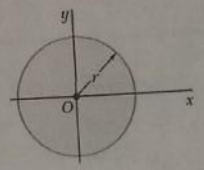
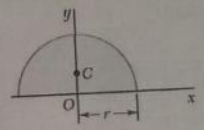
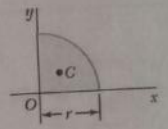
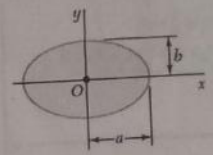
Rectángulo		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triángulo		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Círculo		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicírculo		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Cuarto de círculo		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Elipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3 b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

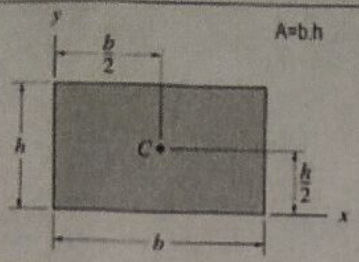
Figura 9.12 Momentos de inercia de formas geométricas comunes.

Es necesario señalar que el radio de giro de un área compuesta *no* es igual a la suma de los radios de giro de las áreas componentes. Para determinar el radio de giro de un área compuesta, es necesario que primero se calcule el momento de inercia del área.

www.FreeLibros.me

Área momento de inercia

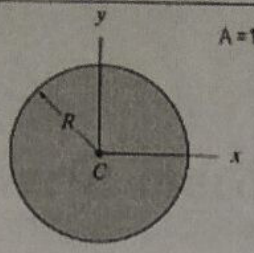
Rectángulo



$$\bar{I}_x = \frac{bh^3}{12} \quad \bar{I}_y = \frac{b^3h}{12} \quad \bar{I}_{xy} = 0$$

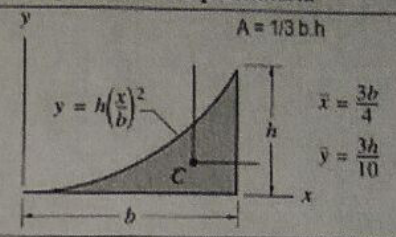
$$I_x = \frac{bh^3}{3} \quad I_y = \frac{b^3h}{3} \quad I_{xy} = \frac{b^2h^2}{4}$$

Círculo



$$I_x = I_y = \frac{\pi R^4}{4} \quad I_{xy} = 0$$

Media parabólica complementaria

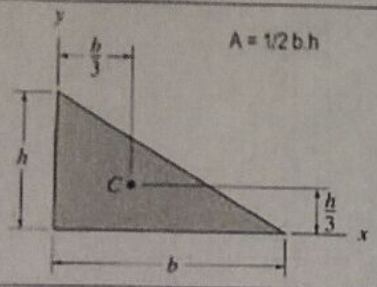


$$\bar{I}_x = \frac{37bh^3}{2100} \quad I_x = \frac{bh^3}{21}$$

$$\bar{I}_y = \frac{b^3h}{80} \quad I_y = \frac{b^3h}{5}$$

$$\bar{I}_{xy} = \frac{b^2h^2}{120} \quad I_{xy} = \frac{b^2h^2}{12}$$

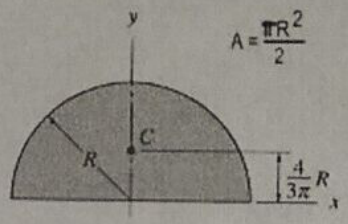
Triángulo rectángulo



$$\bar{I}_x = \frac{bh^3}{36} \quad \bar{I}_y = \frac{b^3h}{36} \quad \bar{I}_{xy} = -\frac{b^2h^2}{72}$$

$$I_x = \frac{bh^3}{12} \quad I_y = \frac{b^3h}{12} \quad I_{xy} = \frac{b^2h^2}{24}$$

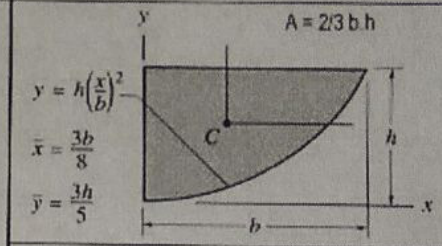
Semicírculo



$$\bar{I}_x = 0.1098R^4 \quad \bar{I}_{xy} = 0$$

$$I_x = I_y = \frac{\pi R^4}{8} \quad I_{xy} = 0$$

Media parábola

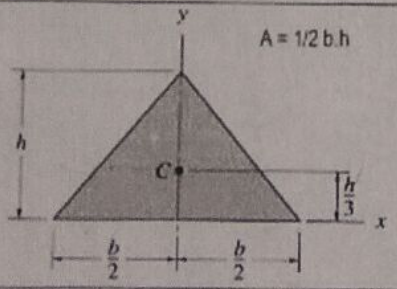


$$\bar{I}_x = \frac{8bh^3}{175} \quad I_x = \frac{2bh^3}{7}$$

$$\bar{I}_y = \frac{19b^3h}{480} \quad I_y = \frac{2b^3h}{15}$$

$$\bar{I}_{xy} = \frac{b^2h^2}{60} \quad I_{xy} = \frac{b^2h^2}{6}$$

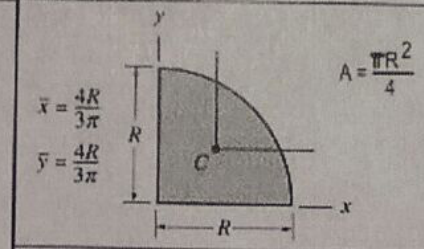
Triángulo isósceles



$$\bar{I}_x = \frac{bh^3}{36} \quad \bar{I}_y = \frac{b^3h}{48} \quad \bar{I}_{xy} = 0$$

$$I_x = \frac{bh^3}{12} \quad I_{xy} = 0$$

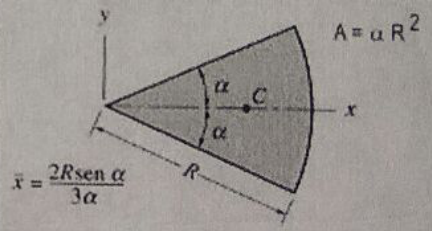
Cuarto de círculo



$$\bar{I}_x = \bar{I}_y = 0.054\,88R^4 \quad I_x = I_y = \frac{\pi R^4}{16}$$

$$\bar{I}_{xy} = -0.016\,47R^4 \quad I_{xy} = \frac{R^4}{8}$$

Sector circular

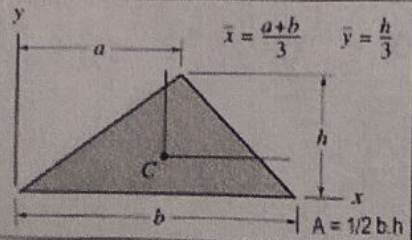


$$I_x = \frac{R^4}{8}(2\alpha - \operatorname{sen} 2\alpha)$$

$$I_y = \frac{R^4}{8}(2\alpha + \operatorname{sen} 2\alpha)$$

$$I_{xy} = 0$$

Triángulo

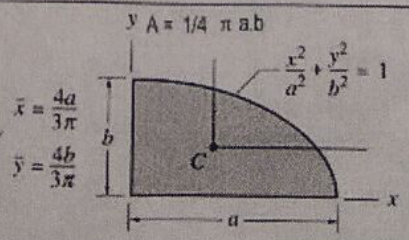


$$\bar{I}_x = \frac{bh^3}{36} \quad I_x = \frac{bh^3}{12}$$

$$\bar{I}_y = \frac{bh}{36}(a^2 - ab + b^2) \quad I_y = \frac{bh}{12}(a^2 + ab + b^2)$$

$$\bar{I}_{xy} = \frac{bh^2}{72}(2a - b) \quad I_{xy} = \frac{bh^2}{24}(2a + b)$$

Cuarto de elipse

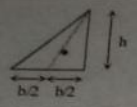
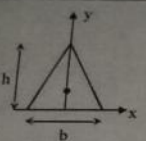
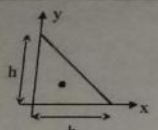
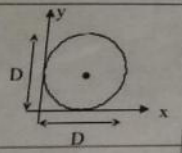
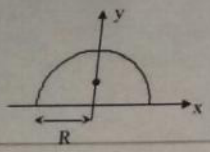


$$\bar{I}_x = 0.054\,88ab^3 \quad I_x = \frac{\pi ab^3}{16}$$

$$\bar{I}_y = 0.054\,88a^3b \quad I_y = \frac{\pi a^3b}{16}$$

$$\bar{I}_{xy} = -0.016\,47a^2b^2 \quad I_{xy} = \frac{a^2b^2}{8}$$

Centróides de Áreas

Triângulo		$\overline{X}$	$\overline{Y}=\frac{h}{3}$ $A=\frac{b\cdot h}{2}$
Triângulo Isósceles/Eqüilátero		$\overline{X}=0$	$\overline{Y}=\frac{h}{3}$ $A=\frac{b\cdot h}{2}$
Triângulo Retângulo		$\overline{X}=\frac{b}{3}$	$\overline{Y}=\frac{h}{3}$ $A=\frac{b\cdot h}{2}$
Círculo		$\overline{X}=\frac{D}{2}$	$\overline{Y}=\frac{D}{2}$ $A=\pi R^2$
Semicírculo		$\overline{X}=0$	$\overline{Y}=\frac{4\cdot R}{3\pi}$ $A=\frac{\pi R^2}{2}$

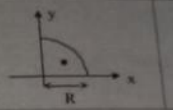
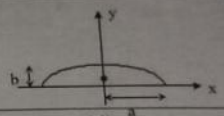
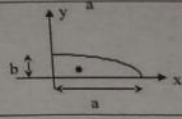
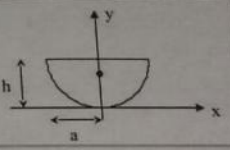
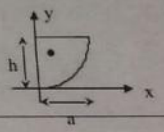
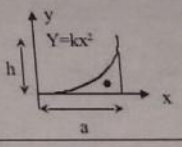
Quarto de Círculo		$\overline{X}=\frac{4\cdot R}{3\pi}$	$\overline{Y}=\frac{4\cdot R}{3\pi}$ $A=\frac{\pi R^2}{4}$
Semi-elipse		$\overline{X}=0$	$\overline{Y}=\frac{4\cdot b}{3\pi}$ $A=\frac{\pi\cdot ab}{2}$
Quarto de elipse		$\overline{X}=\frac{4\cdot a}{3\pi}$	$\overline{Y}=\frac{4\cdot b}{3\pi}$ $A=\frac{\pi\cdot ab}{4}$
Parábola		$\overline{X}=0$	$\overline{Y}=\frac{3h}{5}$ $A=\frac{4\cdot ah}{3}$
Semiparábola		$\overline{X}=\frac{3a}{8}$	$\overline{Y}=\frac{3h}{5}$ $A=\frac{2\cdot ah}{3}$
Arco de Parábola do 2º grau		$\overline{X}=\frac{3a}{4}$	$\overline{Y}=\frac{3h}{10}$ $A=\frac{ah}{3}$

Tabla de Integrales

1. $\int u\,dv = uv - \int v\,du$
2. $\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq 1$
3. $\int \frac{du}{u} = \ln|u| + C$
4. $\int e^u du = e^u + C$
5. $\int a^u du = \frac{a^u}{\ln a} + C$
6. $\int \operatorname{sen} u\,du = -\cos u + C$
7. $\int \cos u\,du = \operatorname{sen} u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \cdot \tan u\,du = \sec u + C$
11. $\int \csc u \cdot \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\operatorname{sen} u| + C$
14. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
15. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
16. $\int \frac{du}{\sqrt{a^2-u^2}} = \operatorname{sen}^{-1}\frac{u}{a} + C$
17. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$
18. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$
19. $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$
20. $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$
21. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$
22. $\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$
23. $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$
24. $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(a+\sqrt{a^2+u^2}\right)$
25. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right)$
26. $\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} + C$
27. $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$
28. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2 u} + C$
29. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$
30. $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\operatorname{sen}^{-1}\frac{u}{a} + C$
31. $\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\operatorname{sen}^{-1}\frac{u}{a} + C$
32. $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$
33. $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{4}\sqrt{a^2-u^2} - \operatorname{sen}^{-1}\frac{u}{a} + C$
34. $\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} - \frac{a^2}{a}\operatorname{sen}^{-1}\frac{u}{a} + C$
35. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$
36. $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2 u}\sqrt{a^2-u^2} + C$
37. $\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\operatorname{sen}^{-1}\frac{u}{a} + C$
38. $\int \frac{du}{(a^2-u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}} + C$
39. $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$
40. $\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$
41. $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$
42. $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$
43. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$
44. $\int \frac{u^2\,du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$
45. $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2 u} + C$
46. $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$
47. $\int \frac{u\,du}{a+bu} = \frac{1}{b^2}(a+bu-a\ln|a+bu|) + C$
48. $\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$
49. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$
50. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$
51. $\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$
52. $\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$
53. $\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left|a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right| + C$
54. $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$
55. $\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$
56. $\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$
57. $\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C$, si $a>0$
$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C$, si $a<0$
58. $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$
59. $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$
60. $\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$
61. $\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$
62. $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$
63. $\int \operatorname{sen}^2 u\,du = \frac{1}{2}u - \frac{1}{4}\operatorname{sen} 2u + C$
64. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\operatorname{sen} 2u + C$
65. $\int \tan^2 u\,du = \tan u - u + C$
66. $\int \cot^2 u\,du = -\cot u - u + C$
67. $\int \operatorname{sen}^3 du = -\frac{1}{3}(2+\operatorname{sen}^2 u)\cos u + C$
68. $\int \cos^3 u\,du = \frac{1}{3}(2+\cos^2 u)\operatorname{sen} u + C$
69. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$
70. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\operatorname{sen} u| + C$
71. $\int \sec^3 u\,du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$
72. $\int \csc^3 u\,du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u + \cot u| + C$
73. $\int \operatorname{sen}^n u\,du = \frac{1}{n}\operatorname{sen}^{n-1} u\cos u + \frac{n-1}{n}\int \operatorname{sen}^{n-2} u\,du$
74. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\operatorname{sen} u + \frac{n-1}{n}\int \cos^{n-2} u\,du$
75. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$
76. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$
77. $\int \sec^n u\,du = \frac{1}{n-1}\tan u\,\sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$
78. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u\,\csc^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$
79. $\int \operatorname{sen} au\operatorname{sen} bu\,du = \frac{\operatorname{sen}(a-b)u}{2(a-b)} + \frac{\operatorname{sen}(a+b)u}{2(a+b)} + C$
80. $\int \cos au\cos bu\,du = \frac{\operatorname{sen}(a-b)u}{2(a-b)} - \frac{\operatorname{sen}(a+b)u}{2(a+b)} + C$
81. $\int \operatorname{sen} au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$
82. $\int u\operatorname{sen} u\,du = \operatorname{sen} u - u\cos u + C$
83. $\int u\cos u\,du = \cos u + u\operatorname{sen} u + C$
84. $\int u^n \operatorname{sen} u\,du = -u^n\cos u + n\int u^{n-1}\cdot\cos u\,du$
85. $\int u^n\cos u\,du = u^n\operatorname{sen} u - n\int u^{n-1}\cdot\operatorname{sen} u\,du$
86. $\int \operatorname{sen}^n u\cdot\cos^m u\,du = $ (2 result. equivalentes)
$= -\frac{\operatorname{sen}^{n-1}\cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \operatorname{sen}^{n-2} u\cdot\cos^m u\,du$
$= \frac{\operatorname{sen}^{n+1}\cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \operatorname{sen}^n u\cdot\cos^{m-2} u\,du$
87. $\int \operatorname{sen}^{-1} u\,du = u\operatorname{sen}^{-1} + \sqrt{1-u^2} + C$
88. $\int \cos^{-1} u\,du = u\cos^{-1} - \sqrt{1-u^2} + C$
89. $\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$
90. $\int u\operatorname{sen}^{-1} u\,du = \frac{2u^2-1}{4}\operatorname{sen}^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$
91. $\int u\cos^{-1} u\,du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$
92. $\int u\tan^{-1} u\,du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$
93. $\int u^n \operatorname{sen}^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\operatorname{sen}^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right]$, $n \neq -1$
94. $\int u^n\cos^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right]$, $n \neq -1$